SyDe 312 - Numerical Methods Unit I Linear Systems

Assignment ♯4: Singular value decomposition problems

1. **Graphical illustration of eigenvectors.** Generate the matrix $A = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ in the MATLAB command window (notice that A is a symmetric matrix). Then type the command: `eigshow(A)` . A graphics window should open. The matrix A should be shown above the graph (in MATLAB notation). Move x around the circle with the cursor and search for the special directions of x where Ax and x lie on a straight line. When x points in one of these directions, it is an eigenvector of the matrix A and $Ax = \lambda x$, with λ the corresponding eigenvalue of A. Since x is a unit vector, the length of Ax is$|\lambda|$. If Ax points in the same direction as x, then $\lambda > 0$, otherwise it is negative. Find the two directions x so that Ax is parallel to x. Give a rough estimate (magnitude and sign) for the eigenvalues of A based on the graph (remember that x is a unit vector).

   Now use MATLAB to calculate eigenvalues and eigenvectors of A by the command: `[Q, E] = eig(A)` (the columns of Q are the normalized eigenvectors and the diagonal entries of E are the eigenvalues). Compare the calculated eigenvalues with your graphical estimates for the eigenvalues. Use MATLAB to verify that Q is an orthogonal matrix. What does this tell you about the angle between the two eigenvectors? Verify that $A = QEQ^T$. Close the eigshow window.

2. **Graphical illustration of the SVD.** Generate a random 2×2 matrix A using the MATLAB command `A= rand(2,2)`. Then type `eigshow(A)` to open the graphics window. Click on the svd button on the right side of the window. Your matrix A should appear (in MATLAB notation) in the menu bar above the graph. The graph shows a pair of orthogonal unit vectors x and y, together with the transformed vectors Ax and Ay.

   Move the pointer onto the vector x, and then make the pair of vectors x and y go around a circle. The transformed vectors Ax and Ay then move around an ellipse, but generally Ax is not perpendicular to Ay. Search for the position of x and y so that Ax is perpendicular to Ay. When this happens, then the singular values σ_1 and σ_2 of A are the lengths of the vectors Ax and Ay. Give a rough estimate of these lengths from the graph (remember that x and y have length one).

   Use MATLAB to calculate the singular value decomposition of A by: `[U, S, V] = svd(A)`. Verify that $A = USV^T$ (up to numerical roundoff error).

   The diagonal entries of S are the singular values σ_1 and σ_2 of A. Compare the calculated singular values with your graphical estimates. The singular values are the square roots of the eigenvalues of A^TA. Check this by calculating `sqrt(eig(A'*A))`.

   Repeat with other matrices A until you get a feel for the geometric meaning of the SVD.

3. Using only symbolic calculation [i.e. not approximate numerical values] and entirely by hand, construct an SVD for the matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$ following the three steps outlined in the lecture slides.

   Compare your answer to that returned by MATLAB for this matrix.

4. Now that you've done one example entirely by hand, you can use MATLAB as a calculating aid in the remaining problems. You should still do the same procedure as for hand calculation, but use MATLAB to save time and avoid errors in the required routine matrix calculations, finding eigenvalues of A^TA, checking you answers etc.

   Find an SVD for each of the matrices $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ and $A' = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2.1 \end{bmatrix}$

   and compare the two results. Zero the second singular value you got for A', calculate the new USV^T, and compare the result to each of the original matrices.

5. Compute an SVD for each matrix:

   (a) $\begin{bmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{bmatrix}$

   (b) $\begin{bmatrix} 6 & -8 & -4 & 5 & -4 \\ 2 & 7 & -5 & -6 & 4 \\ 0 & -1 & -8 & 2 & 2 \\ -1 & -2 & 4 & 4 & -8 \end{bmatrix}$

6. Compute, and express in the expanded form of lecture slide 92, an SVD for each matrix below. Use the singular values to compute the condition number for each matrix. Zero the smallest singular value, calculate the resulting $A' = USV^T$ matrix, and compare it to the original matrix A. Use the norm $||A' - A||$ to evaluate the difference between the matrices. What are the ranks of the two matrices A and A'? Compare the behaviour of these two as the coefficient matrices of a linear system $Ax = b$ or $A'x = b$, using some suitable b vector. Why is the solution with A' very different from that with A. What can you conclude about the conditioning of the original A matrix?

   (a) $\begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$